

We may, therefore, expect that whenever the viscous effect is great relatively to the hardening effect, a wire will show to greater advantage as regards its breaking strength, and to less advantage as regards its elongation, when tested quickly than when tested slowly. This is probably the case with tin, copper, and other metals in which the viscous effect is great. [I have not tested tin wires, but copper and also brass wires show a perceptible hardening effect under constant stress, much less, however, than that shown by soft iron.]

When, on the contrary, the hardening effect is great relatively to the viscous effect, a wire will show to greater advantage as regards breaking strength, and to less advantage as regards elongation the more slowly it is tested. This is the case with soft iron, at least in circumstances like those of the above experiments.

The experiments show that even such variations in the rate of applying stress, or such intermissions of stress as are liable to occur in practical testing, are sufficient to affect most materially the results of the test.

A batch of specimens cut from the same coil, and similarly treated in every way, give very closely accordant results when tested with the same rate of increment of stress. The examples quoted above are all taken from the same batch of specimens. One peculiarity is common to them and to all the other specimens in the same batch, namely, the double curvature, which appears in the diagrams soon after the limit of elasticity has been passed. On the other hand, other specimens cut from the same bundle of wire, but annealed at another time, do not show this peculiarity.

The examples given may fairly be taken as representative of a large number of tests. I am now extending the experiments; but it appeared to me that the results already arrived at were of sufficient interest to excuse their presentation in this somewhat crude form.

VIII. "Memoir on Abel's Theorem." By R. C. ROWE, Fellow of Trinity College, Cambridge. Communicated by A. CAYLEY, LL.D., F.R.S., Sadlerian Professor of Pure Mathematics in the University of Cambridge. Received May 27, 1880.

(Abstract.)

The object of this paper is to present in a shortened and simplified form the processes and the results of Abel's famous memoir, "Sur une propriété générale d'une classe très-étendue de fonctions transcendentes," composed and offered to the French Institute in 1826, but first published in the "Mémoires des Savans Étrangers" for 1841.

The generality and the power of this memoir are well known, but its form is not attractive. Boole, indeed, in a paper on a kindred subject (Phil. Trans. for 1857, pp. 745—803) says, “As presented in the writings of Abel . . . the doctrine of the comparison of transcendents is repulsive, from the complexity of the formulæ in which its general conclusions are embodied.” Boole’s theorems, however, escape this charge only with loss of the generality which makes Abel’s valuable.

But this complexity is rather apparent than fundamental. It is attempted, by rearrangement of parts, by separation of essential from non-essential steps, by changes of notation, in particular by the introduction of a symbol and a theorem discussed by Boole in the paper already referred to, and by the addition of examples of the processes and results, to reduce this part of an important subject to a shape more simple, while no less general, than the original.

Of the three sections which compose the paper the first contains the discussion of the main question,

Is it always possible to establish between the values for different variables of the integral of an algebraic function however complex, algebraic relations, the variables themselves being connected by any requisite algebraic laws?

If, for example, $\int X dx = F(x)$,

where X is an algebraic function of x , rational or irrational, integral or fractional, is it necessarily possible by connecting $x_1, x_2, \dots x_n$ by any requisite algebraic laws to obtain an algebraic (or logarithmic) expression for the sum $F(x_1) + F(x_2) + \dots + F(x_n)$?

This question is suggested on the one hand by such well-known results as

$$F(x_1) + F(x_2) = \text{constant, where } X = \frac{1}{\sqrt{1-x^2}}, \text{ if } x_1^2 + x_2^2 = 1,$$

$$\text{and } F(x_1) + F(x_2) + F(x_3) = 0 \text{ where } X = \frac{1}{\sqrt{1-x^2} \cdot 1-k^2x^2}$$

$$\text{if } (1-x_1^2)(1-x_2^2)(1-x_3^2) = (2-x_1^2-x_2^2-x_3^2+k^2x_1^2x_2^2x_3^2)^2;$$

and on the other hand by the possibility of finding algebraical expressions for many symmetric functions of the roots of equations, though these roots may not be separately determinable.

This combination of the theory of integrals and the theory of equations furnishes in fact the key to the problem, as enabling us to express the “requisite algebraic laws” very concisely by a *single* equation of which the variables are roots, and whose coefficients are not independent but connected by a corresponding number of relations.

The notation is this :—

We first, to escape the inconvenience of irrational and fractional forms, introduce two functions, f, χ , where X , the subject of integration $\equiv f(x, y)$, y being determined as a function of x by the equation

$$\chi(x, y) \equiv y^n + p_1 y^{n-1} + \dots + p_{n-1} y + p_n = 0,$$

where $p_1, p_2 \dots p_n$ are rational integral functions of x , and χ therefore a rational integral function of x, y ; while f is rational, but not necessarily integral. This assumption will cover all cases.

[For example, in the case $X = \frac{1}{\sqrt{1-x^2}}$,

we have the equations $f(x, y) \equiv \frac{1}{y}$,

$$\chi(x, y) \equiv y^2 - (1 - x^2) = 0].$$

We next change $f(x, y)$ so that its denominator shall be the product of $\chi'(y)$, the differential coefficient of χ with respect to y , and a function of x only, and write

$$f(x, y) \equiv \frac{f_1(x, y)}{f_2(x) \cdot \chi'(y)}.$$

This is shown to be always possible.

We then introduce the equation whose roots are the variables of the functions we add. This equation is obviously not arbitrary. It is necessary and sufficient to take the result of eliminating y between χ and any other function of x, y which by the use of χ can be made of (at most) the $n-1$ th degree in y .

Let this function be

$$\theta(y) \equiv q_{n-1} y^{n-1} + \dots + q_1 y + q_0,$$

and let the result of elimination be called E .

Here $q_{n-1}, q_{n-2} \dots q_1, q_0$ are rational integral functions of x , and E is a function of x and of their coefficients.

It may happen that, owing to relations between these coefficients, the equation $E=0$ is satisfied by values of x independent of them. Let $F_0(x)$ be the product of the corresponding factors, and write

$$E(x) = F_0(x) F(x).$$

It only remains to define a symbol Θ , used by Boole in his paper on the comparison of transcendents. The following is his definition :—
“If $\phi(x)f(x)$ be any function of x composed of two factors $\phi(x), f(x)$, whereof $\phi(x)$ is rational, let

$$\Theta[\phi(x)]f(x)$$

denote the result obtained by successively developing the function in

ascending powers of each simple factor $x-a$ in the denominator of $\phi(x)$, taking in each development the coefficient of $\frac{1}{x-a}$, adding together the coefficients thus obtained, and subtracting from the result the coefficient of $\frac{1}{x}$ in the development of the same function $\phi(x)f(x)$ in descending powers of x .*

Then the theorem of this section is

$$\Sigma \int f(x, y) dx = \Theta \left[\frac{1}{f_2(x)F_0(x)} \right] F_0(x) \Sigma \frac{f_1(x, y)}{\chi'(y)} \log \theta(y) + C.$$

This formula is more general than either of Boole's, while it seems more concise and intelligible than that of Abel to which it corresponds.

The rest of this section is occupied with the application of the theorem to several simple cases. Those have been chosen of which the results were well known (*e.g.*, the circular and elliptic functions), with a view to the comparison of method.

In the second section it is shown to follow from the results of Section I that the sum of any number of integrals of the form considered may be expressed in terms of a number of such integrals equal to the number of conditions necessarily connecting the variables in the proposition already discussed; and the question is investigated.

What is the least value of which this number of conditions is susceptible?

It is proved independent of everything but the form of the equation $\chi(x, y) = 0$ by which y is determined as a function of x ; the answer being that if this equation has $n_1\mu_1$ roots of the form $y = Cx^{\frac{m_1}{\mu_1}}$, $n_2\mu_2$ of the form $Cx^{\frac{m_2}{\mu_2}}$, and so on, then this least number is

$$\sum_{s \geq r} n_r m_r n_s \mu_s + \frac{1}{2} \Sigma n^2 m \mu - \frac{1}{2} \Sigma n m - \frac{1}{2} \Sigma n - \frac{1}{2} n + 1.$$

The expressions for the roots are according to descending powers of x ; each $\frac{m}{\mu}$ is a fraction in its lowest terms with denominator positive; the series $\frac{m_1}{\mu_1}$, $\frac{m_2}{\mu_2}$, etc., are in descending order of magnitude; and this equality of powers in sets is shown to subsist.

This section concludes with an example of its processes.

The third section contains three distinct parts: first, a generalization of the theorem of Section I, showing that a similar expression to that

* Cauchy had used in his "Calcul des Résidus" a symbol which only differs from this Θ in the absence of the subtractive term last mentioned.

obtained there may be found for the sum of any number of such functions each multiplied by any rational number, positive or negative, integral or fractional; secondly, an investigation of the conditions that the algebraic expression obtained for the sum of the integrals considered in Section I may reduce to a constant; and lastly, a discussion, as an example of all the results of the paper, of the case

$$X = \frac{\phi}{\psi^{\frac{m}{n}}}$$

where $\phi = \frac{\phi_1(x)}{\phi_2(x)}$, ϕ_1 and ϕ_2 being rational integral functions, while y is a root of the equation

$$\chi \equiv y^n - \psi = 0,$$

and $\theta \equiv q_{n-1}y^{n-1} + q_{n-2}y^{n-2} + \dots + q_1y + q_0$.

A considerable simplification is introduced into the second part of this section by placing it after (instead of, with Abel, before) Section II.

In the Appendix will be found notes on three points in the paper; and a list—it is hoped complete—of the errata in Abel's memoir.

There follows an addition from Professor Cayley, wherein it is shown that the foregoing expression for the least value of the number of conditions is equal to the deficiency (*Geschlecht*) of the curve represented by the equation $\chi(x, y) = 0$. That this least value is equal to the deficiency is a leading result in Riemann's theory of the Abelian integrals; the before-mentioned assumptions as to the form of the roots of the equation $\chi(x, y) = 0$ considered as an equation for the determination of y are equivalent to the assumption that the curve $\chi(x, y) = 0$ has certain singularities; and it is in the addition shown that the resulting value of the deficiency, as calculated by the formulæ in Professor Cayley's paper "On the Higher Singularities of a Plane Curve," *Quart. Math. Journ.*, vol. vii (1866), pp. 212–222, has in fact the foregoing value.